

NOTE

ON THE EXISTENCE OF HAMILTONIAN CYCLES IN A CLASS OF RANDOM GRAPHS

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A digraph with n vertices and fixed outdegree m is generated randomly so that each such digraph is equally likely to be chosen. We consider the probability of the existence of a Hamiltonian cycle in the graph obtained by ignoring arc orientation. We show that there exists m (≤ 23) such that a Hamiltonian cycle exists with probability tending to 1 as n tends to infinity.

1. Introduction

In this paper digraphs do not have loops or repeated arcs.

For a digraph D let GD be the graph obtained by replacing each directed arc (v, w) by an undirected edge $\{v, w\}$.

For positive integers m and n with $m < n$, let $\mathcal{D}(m, n)$ be the set of all vertex-labelled digraphs with n vertices and mn arcs such that each vertex has outdegree m .

Let $M = |\mathcal{D}(m, n)| = \binom{n-1}{m}^n$. We consider the following problem: If D is chosen at random from $\mathcal{D}(m, n)$ so that each such digraph has probability $1/M$ of being chosen, what is the probability that GD has a Hamiltonian cycle?

The main result of this paper is that there exists $m_0 \leq 23$ such that $\lim_{n \rightarrow \infty} \text{Prob}(GD \text{ is Hamiltonian}) = 1$ if and only if $m \geq m_0$.

One motivation for looking at this problem is that when a random graph is chosen by choosing edges independently with the same probability, Hamiltonian cycles appear (in a probabilistic sense) at the same time that the minimum vertex degree reaches 2 (Komlós and Szemerédi [2]). This requires about $\frac{1}{2}n \log n + n \log \log n$ edges, and it is of interest to try and reduce this number by ensuring, in some way, that each vertex has at least a certain degree.

In a previous paper [1] we studied the probable connectivity of these graphs; work on this was stimulated by Walkup's results on random regular bipartite digraphs [5].

2. Main result

Notation. $V_n = \{1, \dots, n\}$. For $\alpha > 0$, $V(\alpha, n) = \{S \subseteq V_n : |S| \leq \alpha n\}$.

For a digraph D with vertex set V and arc set A , we define, for $S \subseteq V$, $\delta_D^+(S) = \{w \in V - S : \text{there exists } v \in S \text{ such that } (v, w) \in A\}$.

For a graph G with vertex set V and edge set E , we define, for $S \subseteq V$, $\delta_G(S) = \{w \in V - S : \text{there exists } v \in S \text{ such that } \{v, w\} \in E\}$.

Lemma 2.1 [1]. *If $m \geq 2$ and $C(m, n) = \{D \in \mathcal{D}(m, n) : GD \text{ is connected}\}$, then*

$$\lim_{n \rightarrow \infty} \text{Prob}(D \in C(m, n)) = 1.$$

By $\text{Prob}(D \in C(m, n))$ we mean $|C(m, n)|/|\mathcal{D}(m, n)|$.

Suppose now $P = (v_1, \dots, v_k)$ is a longest path in a graph $G = (V, E)$. If $t \neq k - 1$ and $\{v_k, v_t\} \in E$, then $P' = (v_1, \dots, v_t, v_k, v_{k-1}, \dots, v_{t+1})$ is also a longest path of G . If $s \neq t, t + 2$ and $\{v_{t+1}, v_s\} \in E$, we can create another longest path P'' using a similar 'flip'.

Keeping v_1 fixed, let $EP(v_1)$ be the set of other endpoints of longest paths formed by doing all possible sequences of flips.

Lemma 2.2 (Pósa [4]). *If $w \in P - EP(v_1)$, then w is adjacent to a vertex of $EP(v_1)$ in G if and only if w is adjacent to some vertex of $EP(v_1)$ on P .*

Corollary 2.3. $|\delta_G(EP(v_1))| \leq 2|EP(v_1)| - 1$.

Remark 2.4. We can of course take each $v \in EP(v_1)$ and use this as a fixed endpoint to create another set of endpoints $EP(v)$ satisfying Corollary 2.3.

Lemma 2.5. *Let $A(\alpha, m, n) = \{D \in \mathcal{D}(m, n) : S \in V(\alpha, n) \text{ implies } |\delta_D^+(S)| \geq 3|S|\}$. Then, for any $\alpha < \frac{1}{4}$, there exists $m(\alpha)$ such that, for $m \geq m(\alpha)$,*

$$\text{Prob}(D \in \mathcal{D}(m, n) - A(\alpha, m, n)) = O(1/n).$$

Proof. If $D \in \mathcal{D}(m, n) - A(\alpha, m, n)$, then there exists $S \subseteq V_n$ with $|S| \leq \alpha n$, and $T \subseteq V_n - S$ with $|T| = 3|S| - 1$ such that $\delta_D^+(S) \subseteq T$. The probability of this event is

clearly bounded above by

$$\begin{aligned} & \sum_{k=\lceil (m+2)/4 \rceil}^{\lfloor \alpha n \rfloor} \frac{n!}{k! (3k-1)! (n-4k+1)!} \left(\frac{\binom{4k-2}{m}}{\binom{n-1}{m}} \right)^k \\ & \leq \frac{1}{2\pi} \sum_k \frac{3(1+o(1))k}{n-4k+1} \left(\frac{n}{3k^2(n-4k)} \right)^{1/2} \frac{n^n}{k^k (3k)^{3k} (n-4k)^{n-4k}} \left(\frac{4k}{n} \right)^{km} \\ & = \frac{1}{2\pi} \sum_k \left(\frac{3(1+o(1))n}{(n-4k)^3} \right)^{1/2} \left(\left(\frac{4k}{n} \right)^{m-4} \frac{256}{27} \right)^k \left(\frac{n-4k}{n} \right)^{4k-n}. \end{aligned}$$

Let m be such that

$$\varepsilon = m - 4 - \sup_{0 < x \leq \alpha} \left(\frac{(1-4x)\log(1-4x) - x \log(256/27)}{x \log 4x} \right) > 0.$$

Then for $1 \leq k \leq \alpha n$ we have

$$\left(\left(\frac{4k}{n} \right)^{m-4} \frac{256}{27} \right)^k \left(\frac{n-4k}{n} \right)^{4k-n} \leq \left(\frac{4k}{n} \right)^{\varepsilon k}.$$

Thus the probability in question is bounded above by

$$\frac{1}{2\pi} \sum_{k=1}^{\alpha n} \left(\frac{3(1+o(1))n}{(n-4k)^3} \right)^{1/2} \left(\frac{4k}{n} \right)^{\varepsilon k} = O(n^{-(1+\varepsilon)}). \quad \square$$

Definition. A graph G has *property LC* if a longest cycle of G has the same number of vertices as a longest path of G .

Lemma 2.6. Let $B = B(m, n) = \{D \in \mathcal{D}(m, n) : GD \text{ does not have Property LC}\}$. For $\alpha > 0$ and $m \geq m(\alpha)$,

$$\text{Prob}(D \in B) \leq ((m/(m-2))^{1/2} (1-\alpha)^\alpha)^n + O(1/n).$$

Proof. Let $\mathcal{D}(m, n) = \{D_1, \dots, D_M\}$ and suppose that $A = A(\alpha, m, n) = \{D_1, \dots, D_{M'}\}$. It follows from Lemma 2.5 that $1 - M'/M = O(1/n)$.

Given $D_i \in A$, construct $N = m^n$ coloured digraphs D_{i1}, \dots, D_{iN} as follows: for each vertex v of D_i choose one arc leaving v and colour it green; colour the remaining arcs blue.

We note that, for $D_i \in A$, the blue subdigraph Δ_{ij} of D_{ij} satisfies

$$S \in V(\alpha, n) \text{ implies } |\delta_\Delta^+(S)| \geq 2|S|, \text{ where } \Delta = \Delta_{ij}. \quad (2.1)$$

Next let $a_{ij} = 1$ if no arc joins 2 endpoints of a longest path of $G\Delta_{ij}$; otherwise $a_{ij} = 0$.

We show next that if $D_i \in B$, then

$$\sum_{j=1}^N a_{ij} \geq N_1 = ((m-2)/m)^{n^*} N. \quad (2.2)$$

Suppose then that $D_i \in B$ and GD_i has a longest path P with k vertices. At least one colouring t yields a GD_{it} in which the edges of P are blue. Clearly $a_{it} = 1$ since $D_i \in B$.

Now in D_{it} fix $k-1$ arcs Q that together produce P (there may be some choice here). In the subdigraph induced by these $k-1$ arcs, let there be k_i vertices of outdegree $i = 0, 1, 2$.

Now, for any colouring j of D_i in which the arcs in Q are coloured blue, $a_{ij} = 1$. Since there are

$$m^{n-k} \prod_{i=0}^2 (m-i)^{k_i} \geq m^{n-k/2} (m-2)^{k/2}$$

such colourings, (2.2) follows immediately.

Thus, if $M_1 = |A \cap B|$,

$$M_1 \leq \sum_{i=1}^{M'} \sum_{j=1}^N a_{ij} / N_1 \quad (2.3)$$

To bound the double sum we construct the following partition: for $\Delta \in \mathcal{D}(m-1, n)$, let $X_\Delta = \{D_{ij} : \Delta_{ij} = \Delta\}$. Let $N_\Delta = |\{D_{ij} \in X_\Delta : i \leq M', a_{ij} = 1\}|$. We shall show next that

$$N_\Delta \leq (1-\alpha)^{\alpha n} (n-m)^n \quad \text{for all } \Delta. \quad (2.4)$$

Thus

$$\begin{aligned} \sum_{i=1}^{M'} \sum_{j=1}^N a_{ij} &= \sum_{\Delta \in \mathcal{D}(m-1, n)} N_\Delta \\ &\leq (1-\alpha)^{\alpha n} (n-m)^n |\mathcal{D}(m-1, n)| = (1-\alpha)^{\alpha n} MN. \end{aligned}$$

Then, from (2.2) and (2.3), $M_1 \leq (1-\alpha)^{\alpha n} (m/(m-2))^{n/2} M$. The result now follows as $|B| \leq M_1 + (M - M')$.

To prove (2.4), select a particular Δ . Let $P = (v_1, \dots, v_k)$ be some longest path of $G\Delta$. Let $EP = EP(v_1) \cup \{v_1\}$ and for $v \in EP$ let $EP(v)$ be defined as in Remark 2.4.

It follows from Corollary 2.3 and (2.1) that, if $N_\Delta > 0$, $|EP| \geq \alpha n$ and $|EP(v)| \geq \alpha n$ for all $v \in EP$.

Now consider all ways of adding 1 new green arc to each vertex of Δ . There are $(n-m)^n$ ways of doing this and of these no more than $(1-\alpha)^{\alpha n} (n-m)^n$ ways avoid joining some $v \in EP$ to some $v \in EP(v)$; this is necessary if the coloured digraph constructed is to be a D_{ij} with $a_{ij} = 1$. (2.4) now follows. \square

Using the above lemma, we are now able to prove the main result.

Theorem 2.7. *Let $H(m, n) = \{D \in \mathcal{D}(m, n) : GD \text{ has a Hamiltonian cycle}\}$. There exists m_0 such that, for $m \geq m_0$,*

$$\lim_{n \rightarrow \infty} \text{Prob}(D \in H(m, n)) = 1.$$

Proof. Fix $\alpha > 0$ and $m_0 \geq m(\alpha)$ such that $1 - 2/m_0 > (1 - \alpha)^{2\alpha}$ and choose $D \in \mathcal{D}(m_0, n)$ at random. Suppose that a longest path in GD has k vertices. We know from Lemma 2.6 that, with probability tending to 1, GD has a cycle C with k vertices. Now, if $k < n$, GD is not connected as no vertex of C can be joined to a vertex not in C , since GD has no path with $k + 1$ vertices. But for $m \geq 2$, by Lemma 2.1, the probability that GD is connected tends to 1. Thus $k = n$ with probability tending to 1 and the result follows. \square

Remark 2.8. We know that $m_0 \leq 23$ by taking $\alpha = 0.202$ in Lemmas 2.5 and 2.6.

We conjecture that the smallest value of m_0 is 3.

Remark 2.9. The problem of when D (as opposed to GD) has a Hamiltonian cycle has been solved by McDiarmid [3]; $m = \log n$ is (about) the required value for m .

Using a similar technique, we have recently proved [6] that, for $r \geq 796$, the probability that a random vertex-labelled r -regular graph with n vertices is Hamiltonian tends to 1 as n tends to infinity. Bollobás [7] has obtained a similar result.

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